Mads Tofte, March 1, 1989
Laboratory for Foundations of Computer Science
Department of Computer Science
Edinburgh University

Four Lectures on Standard ML

The following notes give an overview of Standard ML with emphasis placed on the Modules part of the language.

The notes are, to the best of my knowledge, faithful to "The Definition of Standard ML, Version 2"[1], as regards syntax, semantics and terminology. They have been written so as to be independent of any particular implementation. The exercises in the first 3 lectures can be tackled without the use of a machine, although having access to an implementation will no doubt be beneficial. The project in Lecture 4 presupposes access to an implementation of the full language, including modules. (At present, the Edinburgh compiler does not fall into this category; the author used the New Jersey Standard ML compiler.)

Lecture 1 gives an introduction to ML aimed at the reader who is familiar with some programming language but does not know ML. Both the Core Language and the Modules are covered by way of example.

Lecture 2 discusses the use of ML modules in the development of large programs. A useful methodology for programming with functors, signatures and structures is presented.

Lecture 3 gives a fairly detailed account of the static semantics of ML modules, for those who really want to understand the crucial notions of sharing and signature matching.

Lecture 4 presents a one day project intended to give the student an opportunity of modifying a non-trivial piece of software using functors, signatures and structures.

[1] R. Harper, R. Milner and M. Tofte: "The Definition of Standard ML, Version 2", (ECS–LFCS–88–62) Labroatory for Foundations of Computer Science, Dept. of Computer Science, University of Edinburgh.

1 ML at a Glance

Suppose we were to draw a map of the landscape of programming languages. Where would ML fit in? COBOL and ML could safely be put down far apart. The input/output facilities in COBOL operate on specific kinds of input/output devices, for instance allowing the programmer to declare index sequential files. ML just has the notion of STREAMS, a stream being a sequence of characters, much like streams in UNIX or text files in PASCAL. On the other hand, ML is extremely concise compared to the verbose COBOL and ML is much better suited for structuring data and algorithms than COBOL is.

ML is closer related to PASCAL. Like PASCAL, ML has data types and there is a type checker which checks the validity of programs before they are run. Both PASCAL and ML follow the tradition of ALGOL in that variables can have local scope which is determined statically from the source program. However, PASCAL and ML are radically different in how algorithms are expressed. In PASCAL, as in many other languages, a variable can be updated (using `:=`). Algorithms are often expressed as iterated sequences of statements (using `while` loops, for instance), where the effect of executing one statement is to change the underlying store. In ML, statements are replaced by EXPRESSIONS; the effect of evaluating an expression is to produce a value. Moreover, variables cannot be updated; REFERENCES are special values that can be updated, and as all other values they can be bound to identifiers, but only rarely are the values one binds to variables references. Iteration is expressed using recursive functions instead of loops. In ML, functions are values which can be passed as arguments to functions and returned as results from functions, and ML programmers do this all the time. ML is an example of a FUNCTIONAL language; PASCAL is an example of a PROCEDURAL language.

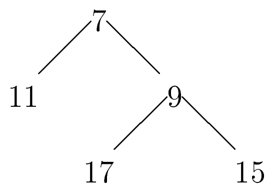
LISP is also sometimes referred to as a functional language. In LISP, programs can be treated as data, so that LISP programs directly can decompose and transform LISP programs. This is harder in ML. On the other hand, the type discipline of ML is extremely helpful in detecting many of the mistakes that pass unnoticed in a LISP program.

Like ADA, ML has language constructs for writing large programs. Roughly speaking, a STRUCTURE in ML corresponds to a PACKAGE in ADA; a SIGNATURE corresponds to a PACKAGE INTERFACE and a FUNCTOR in ML corresponds to a GENERIC PACKAGE in ADA. However, ML admits structures (not just types) as parameters to functors.

1.1 An ML session

An ML session is an interactive dialogue between the ML system and the user. You type a PROGRAM in the form of one or more DECLARATIONS (terminated by semicolon) and the system responds either by accepting the declarations or, in case the program is ill-formed, by printing an error message.

To give a concrete idea about what ML programs look like, we shall work through the following example. Consider the problem of implementing heaps. A HEAP is a binary tree of ITEMS, for example:



For a binary tree to be a heap, it must satisfy that for every item i in the tree, i is less than or equal to all items occurring below i. In the above picture items are integers and the relation "less than or equal" is the normal $\leq$ on integers. The advantage of a heap is that it always gives fast access to a minimal item and that it is easy to insert and delete items from a heap. This has made the heap a popular data structure in a number of very different applications. It was originally conceived under the name "priority queue" as a means of scheduling processes in an operating system; in that case the items are processes and the partial ordering is that process p is less than or equal to process q, if p should be executed no later than q. Heaps are also used in the HEAP SORT algorithm, which is based on the observation that one can sort a list of items by first inserting the items one by one in a heap and then removing them one by one.

1.2 Types and Values

In the following figures we present the ML declarations the author provided in this particular session. The responses from the ML compiler are not shown. For clarity, the actual input has been edited using typewriter font for the reserved words and italics for identifiers, regardless of whether these identifiers are pervasives (e.g. *int*) or declared by the user (e.g. *item*).

```
type item = int;

fun leq(p: item, q: item): bool =
    p <=q;

infix leq;
fun max(p, q) = if p leq q then q else p
and min(p, q) = if p leq q then p else q

datatype tree = L of item
              | N of item * tree * tree;

val t = N(7, L 11, N(9, L 17, L 15));

fun top(L i) = i
|   top(N(i, _, _)) = i;
```

We start out by considering integer heaps only; therefore we first declare the type *item* to be an abbreviation for *int*. Then we declare a function *leq* to be the pervasive `<=` on integers. We then declare that *leq* is to be used as an infix operator, as illustrated in the declaration of the two functions *max* and *min*.

Every binary tree is either a leaf containing an item or it is a node containing an item and two trees (the subtrees). This is expressed by the `datatype` declaration. `datatype` declarations are automatically recursive, i.e. data types can be declared in terms of themselves. This is illustrated by the declaration of *tree*. This data type has two CONSTRUCTORS, L and N. Note that for example `7` is an item, but L APPLIED TO `7`, written L`(7)`, or just L `7`, is of type *tree*. Then the heap from the earlier picture is bound to the value variable t.

Exercise 1 Declare a heap *t'* of the same depth as t containing the integers 78, 34, 5, 12, 15, 28, and 9.

To define a function on trees it will suffice to define its value in the case the argument tree is a node and in the case the tree is a node. The declaration of the function *top* illustrates this. (*top* applied to a tree returns the item at the top of the tree). (*L i*) and (*N*(*i*, _, _)) are examples of PATTERNS. Applying a function (here *top*) to an argument (e.g. *t*) is done by matching the argument against the patterns till a matching pattern is found. For example *top t* evaluates to 7.

1.3 Recursive Functions

```
fun depth(L _) = 1
|   depth(N(i, l, r)) =
        1 + max(depth l, depth r);

depth t;

fun isHeap(L _):bool = true
|   isHeap(N(i, l, r)) =
        i leq top l andalso
        i leq top r andalso
        isHeap l andalso
        isHeap r
```

The function *depth* maps trees to integers; for instance *depth t* evaluates to 3. As spelled out in the declaration of *depth*, the depth of a leaf is 1 and the depth of any other tree is 1 plus the maximum of the depths of the left and right subtrees. The function *depth* is RECURSIVE, i.e. defined in terms of itself. Another example of a recursive function is the function *isHeap* which when applied to a tree returns the value true if the tree is a heap and false otherwise.

Exercise 2 Write a function *size* which when applied to a tree returns the total number of items in the tree.

Exercise 3 The function *top* returns a minimal item of a heap. Write a recursive function *maxItem* which returns a maximal item.

1.4 Raising Exceptions

One often wants to define a function that cannot return a result for some of its argument values. Suppose, for example, that we wish to define a function *initHeap* which for given integer n returns a heap of depth n. This only makes sense for $n \geq 1$. This can be expressed in ML by raising an EXCEPTION in the case $n < 1$. The effect of evaluating the expression `raise` *e*, where *e* is an exception, is to discontinue the current evaluation. Often, the exception will be HANDLED by a `handle` expression (not illustrated by our examples); if no handler catches the exception, it propagates to the top-level where it will be reported as an uncaught exception.

```
val initial = 0
exception InitHeap
fun initHeap n =
    if n<1 then raise InitHeap
    else if n = 1 then L(initial)
    else let val t = initHeap(n - 1)
         in  N(initial, t, t)
         end
```

Notice the `let` *dec* `in` *exp* `end` expression. To evaluate it, one first evaluates *initHeap*(*n* - 1) and binds the resulting value to *t*. Then one evaluates the body, *N*(*initial*, *t*, *t*) using this value for *t*. Notice that the scope of the declaration of *t* is the expression

`N(initial, t, t)`; in particular the two occurrences of *t* in that expression do not refer to `N(7, L 11, N(9, L 17, L 15))`.

Exercise 4 Define functions *leftSub* and *rightSub* which when applied to a tree returns the left and the right subtree, respectively.

Finally, we shall write a function *replace* which when applied to a pair `(i, h)`, where *i* is an item and *h* is a heap, returns a pair `(i', h')`, where i' is the item at the top of the heap *h* and h' is a heap obtained from *h* by inserting *i* in place of the top of *h*. We must make sure that the resulting tree really is a heap. Therefore, in the case that *i* is to be inserted in a node above a subtree with a smaller item, *i* swops place with the smaller item.

```
fun replace(i, h) = (top h, insert(i, h))
and insert(i, L _) = L(i)
|   insert(i, N(_, l, r))=
          if i leq min(top l, top r)
          then N(i, l, r)
          else if (top l) leq (top r) then
               N(top l, insert(i, l), r)
          else (* top r < min(i, top l) *)
               N(top r, l, insert(i, r));

val (out1, t1) = replace(10, t);

t;

val (out2, t2) = replace(20, t1)
```

The special parenthesis `(*` and `*)` delimit comments.

Exercise 5 In the case where one recursively inserts *i* in the left subtree, how can one be sure that it is valid to put *top l* above *r* in the tree?

If one types an expression followed by a semicolon (such as *t*; in the above program) the ML system evaluates the expression and prints the result. In the above example, it will turn out that even after we have "replaced" 7 by 10, *t* is bound to the original heap. Indeed, this "replacement" in no way affects the value bound to *t*; it simply results in a new value, which subsequently is bound to *t1*.

Exercise 6 After the last declaration, what values are bound to *out2* and *t2*?

1.5 Structures

The above declarations of heaps and operations on heaps belong together. In ML there is a program unit called a STRUCTURE which encapsulates a sequence of declarations. The following declaration declares a structure *Heap* containing all the declarations (copied from above) encapsulated by `struct` and `end`.

```
structure Heap =
 struct
   type item = int;
   fun leq(p: item: q: item): bool = p <= q;
   fun max(p, q) = ...
   and min(p, q) = ...
   datatype tree = L of item
                 | N of item * tree * tree;
   val t = ...
   fun top(L i) = ...
   fun depth(L _) = ...
   fun isHeap(L _):bool = ...
   val initial = 0
   exception InitHeap
   fun initHeap n = ...
   fun replace(i, h) = ...
   and insert(i, L _) = ...
 end; (* Heap *)

 val smallHeap = Heap.initHeap(1);
 Heap.replace(20, smallHeap);
```

Identifiers declared in a structure are accessed from outside the structure by a LONG IDENTIFIER, for instance *Heap* . *initHeap* (which can be read "the initHeap in Heap" or just "Heap dot initHeap"). In a large program containing many structures, long identifiers make it much easier for the reader to find the definition of an identifier.

1.6 Signatures

The "type" of a structure is called a SIGNATURE. A signature can specify types, without necessarily saying what the types are. Moreover, one can specify values (in particular functions) by specifying a type for each variable, without saying how such a specification can be met by an actual declaration.

```
signature HEAP =
 sig
  type item
  val leq: item * item -> bool
  val max: item * item -> item
  val min: item * item -> item
  datatype tree = L of item
                | N of item * tree * tree
  val t: item
  val top: tree -> item
  val depth: tree -> int
  val isHeap: tree -> bool
  val initial: item
  exception InitHeap
  val initHeap: int -> tree
  val replace: item * tree -> item * tree
  val insert: item * tree -> tree
 end; (* HEAP *)
```

As one can check, the structure *Heap* MATCHES signature *HEAP* in the following sense: for every type specified in *HEAP*, there is a corresponding type in *Heap*; for every exception specified in *HEAP*, there is a corresponding exception in *Heap*; and for every value specified in *HEAP* there is a corresponding value in *Heap* which has the specified type.

1.7 Coersive Signature Matching

However, the signature *HEAP* reflects details of the implementation in *Heap* which heap users should not have to worry about. (Obviously, the value t is completely unnecessary, and there is no reason why users should have access to the constructors L and N given that we have already given the user *initHeap* and *replace*.) By pruning the signature we obtain the following shorter declaration of *HEAP*.

```
signature HEAP =
 sig
  type item
  val leq: item * item -> bool

  type tree
  val top: tree -> item
  exception InitHeap
  val initHeap: int -> tree
  val replace: item * tree -> item * tree
 end; (* HEAP *)
```

This is a much cleaner interface, so whenever we refer to *HEAP* in the following, we mean this version.

In practice, one should write down a signature *before* one attempts to write down a structure which matches it. In this way one can decide what types and operations are needed without having to think about algorithms at the same time. So let us assume that we started out by declaring the *HEAP* signature. We then imprint the view provided by *HEAP* on the declaration of the structure *Heap* by a SIGNATURE CONSTRAINT:

```
structure Heap: HEAP =
struct
  type item = int;
  ...
end; (* Heap *)

? Heap.t
Heap.replace(7, Heap.initHeap 3);
```

After this declaration of *Heap*, we cannot write *Heap.t*, since *t* is not mentioned in *HEAP*. However, we can write *Heap.replace* as *replace* is specified. Moreover, although *HEAP* does not specify that *item* should be *int*, the ML system discovers that *item* is in fact *int* in *Heap* and that is why `7` will be accepted as an *item* in the application *Heap.replace*`(7, `*Heap.initHeap* `3)`. Thus a signature constraint may hide components of a structure, but it does not hide the true identity of the types declared in the structure, except that one can hide the constructors of a `datatype` by specifying it as a `type`.

1.8 Functor Declaration

Almost all of what we did for heaps containing integer items would work for a heap whose items are of a different type. More precisely, given any type *item*, any binary function *leq* on items and any *initial* item, the signature *HEAP* is satisfied by the declarations we have already written. Let us specify the general requirements of a *Heap* structure.

```
signature ITEM =
 sig
  type item
  val leq: item * item -> bool
  val initial: item
 end;
```

What we are after is a structure which is parameterised on any structure, *Item*, say, which matches *ITEM*. In ML, a parameterised structure is called a FUNCTOR. The following table contains the complete functor declaration; the new bits are in bold face.

```
functor Heap(Item: ITEM): HEAP =
struct
 type item = Item.item
 fun leq(p: item, q: item): bool =
      Item.leq(p,q)
 fun intmax(i: int, j) =
      if i <= j then i else j
 infix leq;
 fun max(p, q) = if p leq q then q else p
 and min(p, q) = if p leq q then p else q
 datatype tree = L of item
                | N of item * tree * tree;
 fun top(L i) = i
 |    top(N(i, _, _)) = i;
 fun depth(L _) = 1
 |    depth(N(i, l, r)) =
        1 + intmax(depth l, depth r);
 fun isHeap(L _):bool = true
 |    isHeap(N(i, l, r)) =
        i leq top l andalso
        i leq top r andalso
        isHeap l andalso
        isHeap r
 exception InitHeap
 fun initHeap n =
      if n<1 then raise InitHeap
      else if n = 1 then L(Item.initial)
      else let val t = initHeap(n - 1)
           in  N(Item.initial, t, t)
           end
 fun replace(i, h) = (top h, insert(i, h))
 and insert(i, L _) = L(i)
 |    insert(i, N(_, l, r))=
           if i leq min(top l, top r)
           then N(i, l, r)
           else if (top l) leq (top r) then
                N(top l, insert(i, l), r)
           else (* top r < min(i, top l) *)
                N(top r, l, insert(i, r));
end; (* Heap *)
```

In the first line, *Item* is the PARAMETER structure of the functor and *HEAP* is the RESULT SIGNATURE of the functor. The BODY of the functor is everything after the = in the first line.

Notice that we included declarations of *item* and *leq* in the body of the functor; since the result signature specifies them, they must be provided. If you read the body carefully, you will see that it makes sense for *any* structure which matches *ITEM*.

Exercise 7 Declare a functor *Pair* which takes as a parameter a structure matching the simple signature

```
sig type coord end
```

and has the following result signature:

```
sig
 type point
 val mkPoint: coord * coord -> point
 val x_coord: point -> coord
 val y_coord: point -> coord
end
```

You do not have to name these signatures (by the use of signature declarations); they can be written down directly where you need them, if you prefer.

Exercise 8 When the author first tried to write the *Heap* functor, he simply copied the original depth function which used *max*, not **intmax**. However, the type checker did not let him get away with that. Why?

1.9 Functor Application

We can now get various heaps (indeed heaps of heaps) by applying the *Heap* functor to different argument structures. Of course, we can only apply it to structures that match *ITEM*; this will be checked by the compiler.

Here is how one can get a string heap:

```
structure StringItem =
struct
 type item = string
 fun leq(i:item, j) =
   ord(i) <= ord(j)
 val initial = " "
end;

structure StringHeap = Heap(StringItem)

val (out1, t1) =
     StringHeap.replace("abe",
      StringHeap.initHeap(1));
val (out2, t2) =
     StringHeap.replace("man", t1);
```

The pervasive *ord* function applied to a string *s* returns the ASCII ordinal value of the first character in *s*, and raises exception `Ord` when *s* is empty.

Exercise 9 Declare a structure *IntItem* using the declarations we originally used for integer heaps. Then obtain a structure *IntHeap* by functor application.

Exercise 10 How does one get an integer heap whose top is always *maximal*?

Exercise 11 Declare a structure *IntHeapHeap* whose items themselves are integer heaps. (You can use the *top* function to define a *leq* function on integer heaps.)

1.10 Summary

ML consists of a CORE LANGUAGE and a MODULES LANGUAGE. The core language has values (functions are values), data types, type abbreviations and exceptions. The modules language has structures, signatures and functors. There is no actual language construct called a module, but ML programmers often refer to a module meaning "a structure or a functor".

2 Programming with ML Modules

2.1 Introduction

This lecture gives a more thorough introduction to the modules part of ML and describes a methodology for programming with its main constructs: structures, signatures and functors.

The core language is interactive: you type a declaration, get a reply, type another declaration and so on, thus gradually adding more and more bindings to the top-level environment. If we could think strictly bottom-up, declaring one value or type in terms of the preceding values and types, without ever making unfortunate implementation decisions or losing the perspective of the entire project, then this gradual expansion of the top-level environment would be quite sufficient. Unfortunately, we cannot, indeed a program which is written as one long list of core language declarations can easily end up looking rather like a long shopping list where items have been added in the order they came to mind.

Regardless of whether a programming language is interactive or not, one needs the ability to divide large programs into relatively independent units which can be written, read, compiled and changed in relative isolation from each other.

One approach, taken by some, is to provide more or less language independent software packages that help programmers organise collections of programs typically by allowing (or forcing) them to document their programs in specific ways. The crucial problem with this approach is of course to ensure consistency between the documentation and the programs, in particular to ensure that the information held by the tool really is sufficient to ensure that the constituent units can be put together in a consistent manner.

Another approach, taken in several programming languages (e.g. Ada and ML), is to provide module facilities in the programming language itself. Many of the operations one needs when programming with modules are similar to operations one needs when programming in the small, so many ideas from usual programming languages apply to programming in the large as well. For instance, just as it is a type error (in the small) to add *true* and 7, say, so it is a type error (in the large) to write a module M2, say, assuming the existence of a module M1 which provides a function f, and then combine M2 with an actual module M1 which either does not provide any f or provides an f of the wrong type. The idea is that such mistakes should be detected by a type checker at the modules level.

This leads to the exciting idea of having just one language with constructs that work uniformly for "small" as well as for "large" programs. One such language is Pebble by Burstall and Lampson. In Pebble records can contain types, so a module consisting of a collection of types and values is now itself a value, which for example can be passed as an argument to a function. There are some trade-offs, however. The ML type checker is based on a strict separation of run-time and compile-time. In designing the modules language it has been necessary to restrict the operations on types in comparison with the operations on values in order to maintain the static type checking. This has led to a stratified language, in which the modules language contains phrases from the core language, but not the other way around.

I shall use the term "module" rather vaguely to mean "a relatively independent program unit". In particular languages they have been called "packages", "clusters", "modules" and in ML we use the word "structure".

Likewise, there is no standard terminology for "the type of a module", which has aquired names such as "package description", "interface" and the ML term "signature".

As we shall see, the real power of a modules system comes from the ability to parameterise modules. Ada has "generic packages". ML has "functors".

In ML, a structure is a collection of data types, types, values, exceptions and even other structures. A signature specifies types and data types and gives the types of values and exceptions. A functor is essentially a function from structures to structures. Functors cannot take functors as arguments, nor can they produce functors as results. The purpose of this lecture is to convince you that even this apparently simple notion of functor constitutes a powerful extension of the core language. As will be demonstrated, one can write an entire system using just signatures and functors and then build the system using functor applications.

Imagine we want to write a parser for a programming language. In order to build the parser top-down, we might start by sketching the parser itself. However, as programming normally is a complex process involving both the odd low-level implementation idea and more high-level considerations about overall structure, let us start at an intermediate level, the problem of writing a symbol table. (A symbol table is simply a facility which allows one to store and retrieve information about symbols.)

2.2 Signatures

The way one in ML sketches a structure is to write down a signature. Here is a first sketch of a symbol table signature, called *OTable* as it is opaque in the sense that it does not reveal many implementation details.

```
signature OTable =
sig
  type table
  exception Lookup
  val lookup: table * Sym.sym -> Val.value
  val update: table * Sym.sym * Val.value
              -> table
end
```

At this early stage, we cannot know exactly what symbols are going to be; nor can we know what kind of values we are going to store with the symbols. Therefore we imagine structures *Sym* and *Val* which declare the types *sym* and *value*, respectively. *Sym.sym* is an example of a LONG IDENTIFIER, in this case a long type constructor. The two structure identifiers *Sym* and *Val* are FREE in *OTable*.

There are many different ways of implementing a symbol table which matches this signature. One possibility is to use an association list, i.e. a list of pairs of symbols and values. Since the symbol table is going to be used extensively, we will probably want something more efficient. We cannot use an array, for arrays map integers (rather than symbols) to values. (Actually, the ML language definition does not include arrays, but they are provided in most implementations). But we can implement the symbol table as a hash table: we can require that the *Sym* structure provides a hash function from symbols to integers and then assume the existence of another structure, *IntMap*, which implements maps on the integers. Since the hash function may map different symbols to the same integer, we take an *IntMap* which maps integers to lists of pairs of symbols and values:

```
signature TTable =
sig
 datatype table = TBL of
 (Sym.sym * Val.value) list IntMap.map
 exception Lookup
 val lookup: table * Sym.sym -> Val.value
 val update: table * Sym.sym * Val.value
              -> table
end
```

2.3 Structures

Here is a structure which implements a symbol table.

```
structure SymTbl =
struct
 datatype table = TBL of
 (Sym.sym * Val.value) list IntMap.map
 exception Lookup

 fun find(sym,[ ]) = raise Lookup
 |   find(sym,(sym',v)::rest) =
       if sym = sym' then v
       else find(sym,rest)

 fun lookup(TBL map, s) =
   let val n = Sym.hash(s)
       val l = IntMap.apply(map,n)
   in find(s,l)
   end handle IntMap.NotFound =>
       raise Lookup
 ...
end
```

When binding a structure to a structure identifier one can impose a SIGNATURE CONSTRAINT on the structure.

```
structure SymTbl : OTable =
struct
  ...
end
```

As a result, all identifiers of the structure that are not mentioned in the signature are hidden. In the above example we hide the constructor *TBL* and the function *find*. Besides *update*, the ... in *SymTbl* may declare extra values, exceptions and types, but as a result of the signature constraint, none of these extra components will be visible from outside the structure.

It is often the case that there is not a single signature which "best" constrains a given structure because different parts of the program should see different degrees of details of the structure. (The parser should be written using a opaque signature for the symbol table; by contrast, a structure which prints out the symbol table (for testing, for example) will need to know more details).

During the design of ML it was decided that it is vital to admit different views of the same structure. One way of achieving this is to bind the structure to more that one structure identifier, each time using a different signature constraint.

```
structure SymTbl : TTable =
struct
 datatype table = TBL of
 (Sym.sym * Val.value) list IntMap.map
 exception Lookup
 fun find(sym,[ ]) = raise Lookup
 |    find(sym,(sym',v)::rest) =
       if sym = sym' then v
       else find(sym,rest)

 fun lookup(TBL map, s) =
   let val n = Sym.hash(s)
        val l = IntMap.apply(map,n)
   in find(s,l)
   end handle IntMap.NotFound =>
        raise Lookup
 ...
end

structure SmallTbl: OTable = SymTbl
```

The dynamic evaluation of the `struct`...`end` yields an environment just as if we had typed the constituent declarations at top-level. Dynamically, there is just one *lookup* function, for example, and as a result of the above declarations, this function is shared between *SymTbl* and *SmallTbl*.

Statically, however, the elaboration of the above declarations yields two different views of this environment. Since there is just one *lookup* function, we should of course be free to refer to it as *SymTbl.lookup* or *SmallTbl.lookup*, whichever we prefer. This requires that these two long identifiers have the same type; so the static semantics must be such that the types *SymTbl.table* and *SmallTbl.table* are considered shared.

2.4 Functors

Unfortunately, neither the declaration of the signature *OTable* nor the declaration of the structure *SymTbl* makes sense on its own. The reason is that they both contain free identifiers. *OTable* relies on structures *Val* and *Sym* and *SymTbl* in addition relies on *IntMap*. As a consequence, we can compile neither *OTable* nor *SymTbl* in the initial top-level environment.

What we need to achieve this is clearly the ability to abstract both *OTable* and *SymTbl* on their free identifiers. Such an abstraction is called a FUNCTOR in ML:

```
functor SymTblFct(
 structure IntMap: IntMapSig
 structure Val: ValSig
 structure Sym: SymSig):

sig
 type table
 exception Lookup
 val lookup: table * Sym.sym-> Val.value
 val update: table * Sym.sym * Val.value
                -> table
end =

struct
 datatype table = TBL of
 (Sym.sym * Val.value)list IntMap.map
 exception Lookup

 fun find(sym,[ ]) = raise Lookup
 |    find(sym,(sym',v)::rest) =
       if sym = sym' then v
       else find(sym,rest)

 fun lookup(TBL map, s) =
   let val n = Sym.hash(s)
        val l = IntMap.apply(map,n)
   in find(s,l)
   end handle IntMap.NotFound =>
        raise Lookup
 ...
end
```

Now the *Sym*, *Val* and *IntMap* that occur in the result signature and in the functor body are bound as formal parameters of the functor. Of course for this functor declaration to make sense, we must first declare the three signatures *IntMapSig*, *ValSig* and *SymSig*, but that can be done without worrying about how we get the corresponding structures. Indeed, declaring these signatures is healthy exercise, as it makes us summarise what a symbol table needs to know about symbols, values and intmaps.

Exercise 12 Declare the signatures *IntMapSig*, *ValSig* and *SymSig*. Also complete the functor body by declaring *update*, extending your signatures, if needed.

When, in due course, we have defined actual structures *FastIntMap*, *Data* and *Identifier*, say, corresponding to the formal structures *IntMap*, *Val* and *Sym*, respectively, we can obtain a particular symbol table by applying the symbol table functor.

```
structure MyTbl =
 SymTblFct(structure IntMap = FastIntMap
              structure Val = Data
              structure Sym = Identifier)
```

Dynamically, the functor body is not evaluated when the functor is declared, but once for each time the functor is applied. (In this respect, functors behave a functions in the core language.)

As part of the functor application, the compiler will check to see whether the actual argument structures match the specified signatures. If that is not the case (for instance, if *Identifier* does not contain a type *sym* or *FastIntMap . apply* takes three instead of two arguments) then an error will be reported and hence we are prevented from putting together the inconsistent structures.

What is the signature of *MyTbl*? It is not simply the result signature of *SymTblFct*, for that signature refers to the formal functor parameters *Val* and *Sym*. Clearly, if *Identifier . sym* is *string* and *Data . value* is *real*, then we should be able to write for instance

```
 sqrt(MyTbl.lookup(t, "pi"))
```

The signature of *MyTbl*, therefore, is obtained by substituting the types of the actual arguments for the types in the formal result signature of *SymTblFct*.

```
sig
 type table
 exception Lookup
 val lookup: table * Identifier.sym
               -> Data.value
 val update: table * Identifier.sym
               * Data.value -> table
end
```

2.5 Substructures

As we saw above, the signature of the result of a functor application depends on the actual arguments to the functor. So apparently there is no single signature which describes all the symbol tables which can be created by applying *SymTblFct*. But how, then, are we going to declare a functor *ParseFct*, say, which we can apply to *any* symbol table created by *SymTblFct*?

The solution to this problem is to make explicit in the symbol table signature that any symbol table depends on a *Val* and a *Sym* structure.

```
signature SymTblSig =
sig
 structure Val: ValSig
 structure Sym: SymSig
 type table
 val lookup: table * Sym.sym-> Val.value
 val update: table * Sym.sym * Val.value
               -> table
end
```

The specifications of *Val* and *Sym* no longer refer to particular structures outside the signature, i.e. *Val* and *Sym* are now considered bound in the signature. (Of course the signature identifiers *SymSig* and *ValSig* are still free, but those you have already declared in the exercise.)

The idea is that a structures can contain not just values, exceptions and types as components, but even other structures. These are called the SUBSTRUCTURES of the structure. To make the result of *SymTblFct* match *SymTblSig*, we have to declare structures *Val* and *Sym* in the body. But that is easily done; we simply bind them to the formal parameters.

```
functor SymTblFct(
 structure IntMap: IntMapSig
 structure Val: ValSig
 structure Sym: SymSig) : SymTblSig =
struct
 structure Val = Val
 structure Sym = Sym
datatype table = TBL of
(Sym.sym * Val.value) list IntMap.map
exception Lookup

fun find(sym,[ ]) = raise Lookup
|   find(sym,(sym',v)::rest) =
      if sym = sym' then v
      else find(sym,rest)

fun lookup(TBL map, s) =
  let val n = Sym.hash(s)
       val l = IntMap.apply(map,n)
  in find(s,l)
  end handle IntMap.NotFound =>
       raise Lookup
...
end
```

2.6 Sharing

A signature for lexical analysers might be as follows (a lexical analyser reads individual characters from an input file and assembles them into symbols — in the case of a programming language typically reserved words and identifiers):

```
signature LexSig =
sig
 structure Sym : SymSig
 val getsym : unit -> Sym.sym
end
```

We have included the specification of a substructure *Sym* because a lexical analyser needs to know about symbols. (Indeed, if we want to declare *LexSig* before defining any particular *Sym* structure, we are *forced* to include the substructure specification.

Here is a first attempt at defining *ParseFct*.

```
functor ParseFct(
 structure SymTbl: SymTblSig
 structure Lex: LexSig) =
struct
 ...
  let val next = Lex.getsym()
  in SymTbl.update(table, next, "declared")
  end
end
```

However, the `let` expression in the body is not type correct! Since the type of *getsym*() is *Lex.Sym.sym*, *next* has type *Lex.Sym.sym*. However, by the specification of *update*, its second argument must be of type *SymTbl.Sym.sym*. The problem is that although we have specified that *SymTbl* depends on a *Sym* structure and *Lex* depends on a *Sym* structure, nowhere have we specified that they depend on the **same** *Sym* structure. The type checker will not make an attempt to identify these two types, for the idea is that the functor should be applicable to *any* arguments that satify the formal parameter specification (not just those that satisfy the specification and in addition have extra sharing). Therefore one is allowed to specify needed sharing as well as needed components by a so-called SHARING SPECIFICATION. Grammatically, a sharing specification can occur anywhere amongst structure, type, value and exception specifications.

```
functor ParseFct(
 structure SymTbl: SymTblSig
 structure Lex: LexSig
  sharing SymTbl.Sym = Lex.Sym) =
struct
 ...
  let val next = Lex.getsym()
  in SymTbl.update(table, next, "declared")
  end
end
```

One can specify sharing of structures and of types (but not of values or exceptions). In our example, we have to add yet a sharing specification, this time a type sharing specification.

```
functor ParseFct(
 structure SymTbl: SymTblSig
 structure Lex: LexSig
  sharing SymTbl.Sym = Lex.Sym
  and type SymTbl.Val.value = string) =
struct
 ...
  let val next = Lex.getsym()
  in SymTbl.update(table, next, "declared")
  end
end
```

2.7 Building the System

Notice that we have now written the code of the parser solely by declaring signatures and functors. We have not had to write a single top-level structure declaration. Having finished declaring the parser functor, we can return to the basics and declare functors that implement *Sym* and *Val*. These functors can be NULLARY, i.e. have an empty specification of formal parameters.

Exercise 13 Write nullary functors *ValFct*, *SymFct* and *IntMapFct* whose result match your signatures from the previous exercise.

We can now build the entire system by functor applications and top-level structure declarations.

```
structure Val = ValFct()

structure Sym = SymFct()

structure TTable =
  SymTblFct(structure IntMap=IntMapFct()
            structure Val = Val
            structure Sym = Sym

structure Lex = LexFct(Sym)

structure Parser =
  ParseFct(structure SymTbl = TTable
           structure Lex = Lex)
```

The compiler will check that the sharing specified in the declaration of *ParseFct* really is met by the actual argumets.

Exercise 14 What is wrong with the following attempt to build the parser?

```
structure Val = ValFct()

structure TTable =
  SymTblFct(structure IntMap=IntMapFct()
            structure Val = Val
            structure Sym = SymFct()

structure Lex = LexFct(SymFct())

structure Parser =
  ParseFct(structure SymTbl = TTable
           structure Lex = Lex)
```

2.8 Separate Compilation

Some ML implementations have facilities that allow you to compile declarations, for instance functor declarations, in such a way that the compiled code can persist between sessions. However, even without such a facility, using signatures and functors in the manner described above gives the valuable ability to separately compile modules consisting of signature and functor declarations, although the result of the compilation will not outlive the session.

Most ML systems have a *use* function which allows the ML source to be read from a file rather than from the terminal. One can then keep signatures in suitably named files and use these files at the beginning of each module.

```
use "symb.sig";
use "val.sig";
use "symtbl.sig";
use "lex.sig";
use "parse.sig";

functor ParseFct(
  structure SymTbl: SymTblSig
  structure Lex: LexSig
   sharing SymTbl.Sym = Lex.Sym
   and type SymTbl.Val.value = string) : ParseSig =
struct
  ...
end
```

In this way one avoids repeating the same signature declaration in many files (and thus also the problem of updating all copies if the signature is changed).

2.9 Good Style

It is good practice to keep signatures as small as possible. If one programs using functors and signatures as described above then writing the body of a functor will reveal which components of its formal parameters that particular functor needs to know about.

Different functors will need different details. Rather than gradually extending a single signature till it gets very large, one can use the `include` specification to enrich an existing signature.

```
signature SmallTbl = sig ... end

signature BigTbl =
sig
 include SmallTbl
 datatype DebugInfo = ...
 val printInfo : unit->unit
end
```

2.10 Bad Style

Signature declarations can contain free structure and type identifiers.

Structure declarations can contain free identifiers of any kind.

This allows you to write for example

```
structure Parser: ParseSig= ParseFct(...)
```

Unfortunately it also allows you to write things like

```
structure Parser =
struct
 structure Lex = Lex
 structure MyPervasives = MyPervasives
 structure ErrorReports = ErrorReports
 structure PrintFcns = PrintFcns
 structure Table = Table
 structure BigTable =BigTable
 structure Aux = Aux

 fun f(...) = ...  Table.lookup  ...
end
```

Here, the programmer has apparently made some effort to show that the parser depends on the structures listed at the beginning. However, if he has missed out a couple of structures from his list, it will have no effect on the declarations that follow, and so one does not as a reader feel confident that the list is exhaustive.

Moreover, when the reader wants to find out what the type of the *lookup* function is, he has to look in the declaration of the *Table* structure. In case *Table* is constrained by a signature, the search continues in the declaration of the signature. Otherwise, one will have to look at the code for *lookup*.

Most serious of all, when encountering the call of *lookup* one has no idea whether *lookup* has side effects that are important to other structures. In that case, the value of structuring code into structures and substructures is purely cosmetic. The only reliable help it gives you is a pointer to the structure in which the identifier is declared.

One particular horror is the misuse of `open`. Avaiable both in the core language and in the modules language, `open` *S* is a declaration which has the effect of adding all the bindings of the structure S to the current environment. This is helpful, if one has a single structure *MyPervasives*, say, which is used everywhere in the project. But look at this:

```
structure Parser =
struct
 structure Lex = Lex
 open MyPervasives ErrorReports PrintFcns
       Table BigTable Aux

 fun f(...) = ...  lookup  ...
end
```

Now finding *lookup* is reduced to pure guesswork!

Exercise 15 For each of the above points of criticism, consider to what extent it applies if one programs with signatures and functors only.

3 The Static Semantics of Modules

The purpose of this lecture is to explain the static semantics of modules. In particular, we shall look into the details of the crucial concepts SIGNATURE MATCHING and SHARING.

3.1 Elaboration

Consider the two following signatures, the first of which stem from the *MyTbl* example of Lecture 1.

```
sig
 type table
 exception Lookup
 val lookup: table * Identifier.sym
               -> Data.value
 val update: table * Identifier.sym
               * Data.value -> table
end

sig
 type table
 exception Lookup
 val lookup: table * string -> real
 val update: table * string * real -> table
end
```

In one sense, these signatures are very different; the meaning of the first one depends on the free structures *Data* and *Identifier*, whereas the second depends on the pervasives only. However, if *Identifier.sym* happens to be *string* and *Data.value* happens to be *real* then the two expression are just different ways of expressing the same meaning. In that sense, the two signatures turn out to be equal.

To avoid such confusion concerning equality, it is often helpful to distinguish between a SIGNATURE EXPRESSION (the syntactic object) and a SIGNATURE (its meaning). The transition from signature expressions to signatures is called ELABORATION. We use the word elaboration instead of evaluation, because, unlike evaluation, all elaboration can be done statically, by a compiler. The result of elaborating a signature expression depends on the meaning of the identifiers occurring free in the expression. In any given context, there are infinitely many signature expressions that elaborate to the same signature. It is even the case that in every context, every signature expression elaborates to infinitely many signatures, if it elaborates to any at all. However, among these there will always be some that are PRINCIPAL which means that, in a certain technical sense, all the others are instances of them, and one always takes a principal signature as the meaning of a signature declared at top-level.

Elaboration applies to STRUCTURE EXPRESSIONS and FUNCTOR DECLARATIONS as well, yielding STRUCTURES and FUNCTOR SIGNATURES, respectively.

Essentially, the modules part of ML is a language for computing these (abstract) signatures, structures and functor signatures. The purpose of this lecture is to explain the principles that govern elaboration.

We shall not introduce a separate notation for structures, signatures and functor signatures. In many cases these are very similar to the expressions from which they were obtained, so we make do with the device of "decorating" expressions with so-called names. Names are semantic objects, completely distinct from identifiers; in the above examples, *string* and *Identifier.sym* are both identifiers which elaborate to the same type name.

3.2 Names

```
structure Stack =
struct
 type elt = int
 datatype stack = ST of elt list ref
 val initStack = ST(ref[ ])
end

structure StackUser1 =
struct
 structure Stack1 = Stack
 ...
end

structure StackUser2 =
struct
 structure Stack2 = Stack
 ...
 datatype stack = ST of elt list ref
end
```

All the following sharing equations hold: *StackUser1 . Stack1* = *StackUser2 . Stack2*, *elt* = *int*, *Stack . stack* = *StackUser1 . Stack1 . stack*. None of the following sharing equations hold: *StackUser1* = *StackUser2*, *Stack . stack* = *StackUser2 . stack*

Sharing equations can be decided by decorating programs with NAMES. There are two kinds:

structure names:	$n1, n2, \ldots,$
	$m1, m2, \ldots$
type names:	$t1, t2, \ldots,$
	$s1, s2, \ldots,$
	$unit, int, bool, \rightarrow$

Two structures SHARE if they are decorated by the same structure name; two types SHARE if they are decorated by the same type name.

3.3 Decorating Structures

Each elaboration of a structure expression of the form

`struct ... end`

yields a fresh structure, i.e. a structure decorated by a new name. Therefore such expressions are called GENERATIVE STRUCTURE EXPRESSIONS.

Each elaboration of a data type declaration (`datatype ...`) yields a fresh type i.e., a type decorated by a new name.

```
structure Stack_n1 =
struct
 type elt_int = int
 datatype stack_t1 = ST of elt list ref
 val initStack_t1 = ST(ref[ ])
end

structure StackUser1_n2 =
struct
 structure Stack1_n1 = Stack
 ...
end

structure StackUser2_n38 =
struct
 structure Stack2_n1 = Stack
 ...
 datatype stack_t23 = ST of elt list ref
end
```

To be complete, one would have to decorate each structure not merely by a name but also with its decorated components and subcomponents. (A structure expression in the form of a functor application does not in itself reveal the components of the resulting structure.) However, to keep decorations to a minimum, we shall usually not spell out the decorated subcomponents.

3.4 Decorating Signatures

```
signature StackSig_{(m1,s1,s2)} =
sig_{m1}
 type elt_{s1}
 type stack_{s2}
 val new_{unit→s2} : unit->stack
end

signature TranspSig_{(m1,s1)} =
sig_{m1}
 type elt_{s1}
 type stack_{t1}
  sharing type stack_{t1} = Stack.stack_{t1}
 val new_{unit→t1} : unit->stack
end
```

Bound names are collected at the signature identifier. They are listed between parenthesis to indicate that they are merely place holders.

The bound names of *StackSig* are $m1$, $s1$ and $s2$. The free names of *StackSig* are *unit* and $\rightarrow$. The bound names of *TranspSig* are $m1$ and $s1$. The free names of *TranspSig* are $t1$, *unit* and $\rightarrow$.

Exercise 16 Consider

```
signature Symbol =
sig
 type symbol
 type value
 sharing type value = int
end
```

Decorate this signature declaration with type and structure names. How many bound names are there? How many free?

If two structures are found to share by the static analysis then they really are the same at run-time. Therefore, when decorating signatures one must make sure that if two structures are made to share (by being given the same name) then any type or structure which is visible in both structures must be made to share as well.

Exercise 17 Consider the following signatures most of which you have already seen in Lecture 1.

```
signature ValSig =
sig
 type value
end

signature SymSig =
sig
 eqtype sym
 val hash : sym->int
end

signature LexSig =
sig
 structure Sym : SymSig
 val getsym : unit->Sym.sym
end

signature SymTblSig =
sig
 structure Val: ValSig
 structure Sym: SymSig
 type table
 val lookup:
       table * Sym.sym-> Val.value
 ...
end

signature ParseSig =
sig
 structure Lex : LexSig
 structure Tbl : SymTblSig
  sharing Lex.Sym = Tbl.Sym
 type abstsyn
 val parse : unit->abstsyn
end
```

Decorate these signatures. When one signature refers to another (for instance *LexSig* refers to *SymSig*) you should put a full decoration on the structure identifier (*Sym*), i.e. a decoration which shows both a name and the subcomponents of the structure. Full decorations can be drawn as trees; in the example at hand you can decorate *Sym* by

$$\begin{tikzcd}
 & m1 \arrow[dl, "sym"'] \arrow[dr, "hash"] & \\
s1 & & s1 \to \mathit{int}
\end{tikzcd}$$

Make sure that you decorate shared substructures (for instance *Sym* in *ParseSig*) consistently so as to represent that sharing of two structures implies sharing of their substructures.

3.5 Signature Instantiation

```
structure $Stack_{n1}$ =
struct
 type $elt_{int}$ = int
 datatype $stack_{t1}$ = ST of elt list ref
 fun $new_{unit\to t1}$() = ST(ref[ ])
end

signature $StackSigA_{(m1,s1,s2)}$ =
sig$_{m1}$
 type $elt_{s1}$
 datatype $stack_{s2}$ = ST of elt list ref
 val $new_{unit\to s2}$: unit->stack
end
```

Note that if we substitute $n1$ for $m1$, *int* for $s1$ and $t1$ for $s2$ in the decoration of *StackSigA* then we get the decoaration of *Stack*. We say that *Stack* is an **instance** of *StackSig*. More generally, we say that a structure is an INSTANCE of a signature if the decoration of the former is obtained from the decoration of the latter by performing a substitution of names for the **bound** names of the signature (the free names of the signature must be left unchanged). The process of substituting names for bound names is called REALISATION.

```
structure $Stack_{n1}$ =
struct
 type $elt_{int}$ = $int$
 datatype $stack_{t1}$ = $ST$ of $elt\ list\ ref$
 fun $new_{unit\to t1}$() = $ST$($ref$[ ])
end

signature $StackSigB_{(m1,s1)}$ =
sig$_{m1}$
 type $elt_{s1}$
 datatype $stack_{t1}$ = $ST$ of $elt\ list\ ref$
 sharing type $stack_{t1}$ = $Stack$ .$stack_{t1}$
 val $new_{unit\to t1}$: $unit$->$stack$
end
```

Stack is an instance of *StackSigB* via the realisation $\{m1 \mapsto n1, s1 \mapsto int\}$.

```
structure $OddStr_{n1}$ =
struct
 type $elt_{int}$ = $int$
 val $test_{bool}$ = $false$
end

signature $WrongSig_{(m1,s1)}$ =
sig$_{m1}$
 type $elt_{s1}$
 val $test_{s1}$ : $elt$
end
```

OddStr is not an instance of *WrongSig*, for $s1$ would have to be realised by *int* (because of *elt*) but then *test* is decorated by *int* in the signature and by *bool* in the structure.

3.6 Signature Matching

Matching of a structure against a signature is a combination of two operations. The first, signature instantiation (described above), is concerned with instantiating the bound names of the signature to the "real" names of the structure. The second is concerned with ignoring information in the structure which is not required by the signature.

```
structure $Tree_{n1}$ =
struct
 datatype '$a\ tree_{t1}$ = $LEAF$ of '$a$
            | $NODE$ of '$a\ tree$ * '$a\ tree$
 type $intTree_{int\,t1}$ = $int\ tree$
 fun $max$($a$:$int$, $b$:$int$) =
   if $a > b$ then $a$ else $b$
 fun $depth_{'a\,t1\to int}$($LEAF$ _) = 1
 |     $depth$($NODE$($left$,$right$))=
          $max$($depth\ left$, $depth\ right$)
 end

signature $TreeSig_{(m1,s1,s2)}$ =
sig$_{m1}$
 type '$a\ tree_{s1}$
 type $intTree_{s2}$
 fun $depth_{s2->int}$: $intTree$->$int$
end
```

A structure MATCHES a signature if the structure can be cut down to an instance of the signature by

1. forgetting components;
2. forgetting polymorphism of variables.

Tree matches *TreeSig*. First perform the realisation $\{m1 \mapsto n1, s1 \mapsto t1, s2 \mapsto int\,t1\}$ on the signature. The resulting decoration can be obtained from the decoration of *Tree* by

1. forgetting the constructors *LEAF* and *NODE*
2. instantiating $'a\,t1 \to int$ to $int\,t1 \to int$ (i.e. the realisation of $s2 \to int$)

Exercise 18 Let *mytype* be a type which is declared in a structure and specified in a signature. In which of the following cases can the structure match the signature?

(1) *mytype* is declared as a `datatype` and specified as a `datatype`.

(2) *mytype* is declared as a `datatype` and specified as a `type`;

(3) *mytype* is declared as a `type` and specified as a `type`;

(4) *mytype* is declared as a `type` and specified as a `datatype`.

3.7 Signature Constraints

```
structure Tree: TreeSig =
 struct ... end
```

In a structure declaration with an explicit signature constraint, the resulting view of the declared structure is precisely the one given by the instantiated signature.

In the example above, the resulting view of *Tree* will hide the constructors *NODE* and *LEAF* and the function *max*. Notice, however, that *intTree* is decorated by the instance of $s2$, i.e. by *int* $t1$, where $t1$ is the decoration of *'a tree*. Consequently, *Tree . intTree* and *int Tree . tree* now mean the same thing, namely *int* $t1$. This sharing was obtained through relisation — it was not explicit in *TreeSig*.

In short, explicit signature constraints can remove components and polymorphism, but they do not affect existing sharing.

Here is an example of a structure declared with an explicit signature constraint. You should convince youself that the structure really does match the signature.

```
signature SymSig =
sig
 type sym
 type code
  sharing type code = int
 val hash : sym->int
 val mksym : string->sym
 val nameof : sym->string
end

structure Sym : SymSig =
struct
 datatype sym = SYM of string * int
 type code = int
 fun convert(s: string): code = ...
 fun hash(SYM(s, n))= n
 fun mksym(s) = SYM(s, convert s)
 fun nameof(SYM(s,_))= s
end
```

Exercise 19 Complete the declaration of *convert*.

Exercise 20 Declare a different structure *NewSym*, also constrained by *SymSig*, such that *NewSym . sym* shares with *string*. Which of the following expressions are valid?

(1) `"a" ^ NewSym.mksym "d"`

(2) `"a" ^ Sym.mksym "d"`

3.8 Decorating Functors

Dynamically, the body of a functor is not evaluated when the functor is declared but it is evaluated once for each time the functor is applied.

```
functor StackFct() =
struct
 datatype stack = ST of int list ref
 val data = ST(ref [ ])
 ...
end

structure Stack1 = StackFct()

structure Stack2 = StackFct()
```

Since the two applications of *StackFct* create two distinct references, *Stack1* and *Stack2* are different and must not be seen to share.

Now let us consider the problem of decorating *StackFct* with names. We start out by decorating the body in the usual way. However, each time we need a fresh name, we record it at the $=$ in the first line. The names that hence are accumulated are called the GENERATIVE NAMES of the functor. The generative names are bound in the sense that they stand as place holders for fresh names which we choose when we eventually apply the functor. (Like the bound names in signatures, we write generative names between parenthesis; unlike the bound names of signatures, generative names are written on the right — because they concern the right side only.)

In the case of nullary functors, i.e. functors that take an empty argument, the structure resulting from a functor application is decorated by taking the decoration of the functor body with each generative name replaced by a fresh name.

```
functor StackFct() =(m1,s1)
struct_m1
 datatype stack_s1 = ST of int list ref
 val data_s1 = ST(ref [ ])
 ...
end

structure Stack1_n7 = StackFct()

structure Stack2_n8 = StackFct()
```

Exercise 21 The decorations of *Stack1* and *Stack2* show the top-most structure name only. Complete the decorations.

Notice that *Stack1* and *Stack2* do not share; not even the types *Stack1.stack* and *Stack2.stack* share. Consequently, the variables *Stack1.data* and *Stack2.data* have different types and so the type checker prevents one from mistaking the one for the other.

3.9 External Sharing

Within the body of a functor one may refer to identifiers (of any kind) declared in the context of the functor. Such identifiers are said to occur FREE in the functor. This results in EXTERNAL SHARING, i.e. a decoration in which some of the names stem from outside the functor.

```
structure MyPervasives =
struct$_{n1}$
 datatype num$_{t1}$ = NUM of int
 ...
end

functor StackFct'() =$_{(m2)}$
struct$_{m2}$
 structure MyPer$_{n1}$ = MyPervasives
 type stack$_{t1\ list\ ref}$ =
        MyPer.num list ref
 val data$_{t1\ list\ ref}$:stack = ref [ ]
end

structure Stack1'$_{n8}$ = StackFct'()

structure Stack2'$_{n9}$ = StackFct'()
```

Notice that external names are not generative; they are left unchanged when the functor is applied.

Exercise 22 Which of the following sharing equations hold?
Stack1' = *Stack2'*;
Stack1' . *MyPer* = *Stack2'* . *MyPer*;
type *Stack1'* . *stack* = *Stack2'* . *stack*.

3.10 Functors with Arguments

```
signature SymSig$_{(m1,s1)}$ =
sig$_{m1}$
 eqtype sym$_{s1}$
end

functor SymDir(Sym: SymSig) =$_{(m2,s2)}$
struct$_{m2}$
 datatype dir$_{s2}$ = DIR of
                       Sym.sym->int
 fun update ...
end
```

When decorating the body of a functor which has an argument, we assume that we have a structure (by the name of the formal parameter) which *precisely* matches the parameter signature. We assume neither more components nor more sharing than is specified in the signature, for we want the functor to be applicable to all actual argument structures that match the formal parameter signature.

In the above example we simply assume that the name of *Sym* is $m1$ and that the name of *Sym.sym* is $s1$ (as those names are not used of free structures elsewhere; in general, one might have to rename some of the bound names. Having used $m1$ and $s1$ we simply start generating names from $m2$ and $s2$ in the body.

```
structure Actual$_{n1}$ =
struct type sym$_{string}$ = string
end

structure Result$_{n2}$ = SymDir(Actual)
```

Result receives a fresh structure name and *Result*. *dir* a fresh type name. Note that *Actual* matches *SymSig*.

3.11 Sharing Between Argument and Result

`signature` $SymSig_{(m1,s1)}$ `=`
$\texttt{sig}_{m1}$
 `eqtype` sym_{s1}
`end`

`functor` *SymDir*(*Sym*: *SymSig*) $=_{(m2)}$
$\texttt{struct}_{m2}$
 `type` $dir_{s1\rightarrow int}$ `=` *Sym*. *sym*`->`*int*
 `fun` *update* . . .
`end`

The type name $s1$ is shared between the argument and the body. When the functor is applied, this sharing must be translated into sharing between the actual argument and the actual result.

`structure` $Actual_{n1}$ `=`
`struct type` sym_{string} `=` *string*
`end`

`structure` $Result_{n2}$ `=` *SymDir*(*Actual*)

Exercise 23 Complete the decoration of *Result*.

Exercise 24 (1) Using the latest definition of *SymDir*, is the following expression legal?

`fn` (*d*: *Result*. *dir*) `=>` *d* "abc"

(2) Same question, but for the earlier definition of *SymDir*.

A full decoration of the result of applying a functor with one argument can be obtained as follows:

1. Match the actual argument against the formal parameter signature yielding a realisation which maps bound names to formal signature to names in the actual argument;
2. Apply this realisation to the decoration of the functor body;
3. also substitute fresh names for the generative names of the functor body.

3.12 Explicit Result Signatures

When a functor declaration contains a result signature, the decoration of the functor declaration proceeds as follows:

1. decorate the functor without the result signature;
2. decorate the result signature. If one can get an instance of the result signature by removing components and polymorphicm from the decorated body, then this instance is used as a formal result instead of the decorated body; otherwise the declaration is rejected.

This has the effect that upon application of the functor, sharing is propagated as before, but only the components and polymorphism of the result signature are visible in the actual result.

Exercise 25 Why is it not always the case that obtaining R by

```
functor F(S: SIG): SIG'=
struct...end

structure R = F(...)
```

is equivalent to obtaining R by

```
functor F(S: SIG) =
struct...end

structure R: SIG' = F(...)
```

Give a condition on *SIG'* under which this difference disappears.

4 Implementing an Interpreter in ML

The purpose of this lecture is to show a worked example of program development using ML modules. We shall tackle the problem of implementing a small ML system. The system is of course going to considerable simplified compared to a real ML implementation.

We implement only a few of the language constructs found in real ML. The user of our system will not get the ability to declare new types and data types; however, there will be arithmetic on the build-in integers, `if`...`then`...`else` expressions, and indeed lists, higher order functions and recursion, so it is far from a trivial language. We shall refer to this language as Mini ML.

Moreover, the system will be an interpreter rather than a compiler. It still has a typechecker, indeed we shall see how one can implement a restricted form of polymorphism.

The system is actually running and you can modify and extend it provided you have access to an implementation and to the files listed in Appendix B. To make life easier for you, we provide a parse functor which can parse a string (the Mini ML source expression) into an ABSTRACT SYNTAX TREE, the shape of which will be defined below. The rest of the interpreter works on abstract syntax trees.

The interpreter uses a TYPECHECKER to check the validity of input expressions and an EVALUATOR to evaluate them. Initially, the typechecker and evaluator handle only a tiny subset of Mini ML. In this lecture I shall show how one in successive steps can extend the typechecker to handle polymorphic lists, variables and `let` expressions. In the practical sessions you can extend the evaluator in the same manner (it is easier than extending the typechecker).

The typechecker and the evaluator can be developed independently as long as you do not change the signatures we provide. This will allow you to take the typechecker functors I have written and plug into your own system as you improve the power of your evaluator. Alternatively, you might want to modify or extend my typechecker functors, and take over evaluator functors that other people write.

The source of the bare interpreter is in Appendix A. An overview of how to run the systems is provided in Appendix B.

The development of the typechecker and the evaluator need not be in step. You can disable either by assigning false to one of the variables `tc` and `eval`.

```
signature INTERPRETER=
   sig
      val interpret: string -> string
      val eval: bool ref
      and tc   : bool ref
   end;
```

The syntax of the language is as follows

exp	::=	*exp* `+` *exp*
		exp `-` *exp*
		exp `*` *exp*
		`true`
		`false`
		exp `=` *exp*
		`if` *exp* `then` *exp* `else` *exp*
		exp `::` *exp*
		`[` exp_1 `,` $\cdots$ `,` exp_n `]` $(n \geq 0)$
		`let` x `=` *exp* `in` *exp*
		`let rec` x `=` *exp* `in` *exp*
		x
		`fn` x `=>` *exp*
		exp `(` *exp* `)` (function application)
		n (natural numbers)
		`(` *exp* `)`

The abstract syntax of Mini ML is defined as a datatype in the signature `EXPRESSION`.

Exercise 1   Find this signature. What is the constructor corresponding to `let` expressions?

We program with signatures and functors only. After the signatures, which we shall not yet study, the first functor is the interpreter itself.

Exercise 2   Find this functor. Find the application of `Ty.prType`. Find it's type. What do you think `Ty.prType` is supposed to do? What is the type of `abstsyn`? What do you think the evaluator is supposed to do when asked to evaluate something which has not yet been implemented?

We shall now describe Version 1, the bare typechecker, and then proceed to the extensions.

4.1 VERSION 1: The bare Typechecker (Appendix A)

The first version is just able to type check integer constants and `+`. As signature `TYPE` reveals, the type `Type` of types is abstract, but there are functions we can use to build basic types and decompose them. `unTypeInt` is one of the latter; it is supposed to raise `Type` if applied to any Mini ML type different from the int (however the type int is represented). This is a common way of hiding implementation details, and it might be helpfull to look at how `functor Type` produces a structure which matches the signature `Type`.

As revealed by `signature TYPECHECKER`, the typechecker is going to depend on the abstract syntax and a `Type` structure. However, as you can see from the declaration of `functor TypeChecker`, all the typechecker knows about the implementation of types is what is specified by the signature `TYPE`. This allows us to experiment with the implementation of types to obtain greater efficiency without changing the typechecker, as we shall see in the later stages. As you see from `functor TypeChecker`, all the typechecker is capable of handling is integer constants and `+`.

Exercise 3 Modify the typechecker to handle `true`, `false`, and multiplication of integers.

Given the signature and functor declarations in Appendix A, one can build the system. First we import the parser

```
use "parser.sml";
```

and then we build the system by the following declarations (which can be read from file `build1.sml`).

```
structure Expression= Expression();

structure Parser= Parser(Expression);

structure Value = Value();

structure Evaluator=
   Evaluator(structure Expression= Expression
             structure Value = Value);

structure Ty = Type();

structure TyCh=
  TypeChecker(structure Ex = Expression
              structure Ty = Ty);

structure Interpreter=
  Interpreter(structure Ty= Ty
              structure Value = Value
              structure Parser = Parser
              structure TyCh = TyCh
              structure Evaluator = Evaluator);

open Interpreter;
```

4.2 VERSION 2: Adding lists and polymorphism

The first extension is to implement the type checking of lists. In Version 1 the type of an expression could be inferred either directly (as in the case of `true` and `false`, or from the type of the subexpressions (as in the case of the arithmetic operations). When we introduce list, this is no longer the case. Consider for example the expression

```
if ([] = [9]) then 5 else 7
```

Suppose we want to type check `([] = [9])` by first type checking the left subexpression `[]`, then the right subexpression `[9]` and finally checking that the left and right-hand sides are of the same type before returning the type `bool`. The problem now is that when we try to type check `[]` we cannot know that this empty list is supposed to be an integer list. The typechecker therefore just ascribes the type `'a list` to `[]`, where `'a` is a TYPE VARIABLE. The `[9]` of course turns out to be an `int list`. The typechecker now "compares" the two types `'a list` and `int list` and discovers that they can be made the same by applying the substitution that maps `'a` to `int`. Hence the type of the expression `[]` depends not just on the expression itself, but also on the context of the expression. The context can force the type inferred for the expression to become more specific.

This "comparison" of types performed by the typechecker is called UNIFICATION and is an algebraic operation of great importance in symbolic computing. Indeed, whole programming languages have evolved around the idea of unification (PROLOG, for example). Here is a couple of examples to illustrate how unifications works in the special case of interest, that of unifying types.

```
[ [] , [[5]] ]                                    (1)
```

This expression is well-typed! The point is that the `[]` can be regarded as an `int list list`. Let us see how the typechecker manages to infer the type `int list list list` for (1). The typechecker first rewrites the expression to the equivalent:

```
[] :: (((5 :: []) :: []) :: [])                   (2)
```

Checking the first argument of the topmost `::` yields:

```
[] : 'a1 list                                     (3)
```

To check `(((5 :: []) :: []) :: [])`, we first check the left-hand `((5 :: []) :: [])`. To check this, we first check the left-hand `(5 :: [])`. To check this, we first check the left-hand `5`, for which the typechecker wisely infer the type `int`. Continuing to the right-hand part of `(5 :: [])`, `[]` gets the type `'a2 list`. To check the `::` of `(5 :: [])`, we now unify `int list` and `'a2 list`, which results in the substitution

$$S_1(\texttt{'a2}) = \texttt{int}.$$

Thus the type of `(5 :: [])` is `int list`.

Returning to `((5 :: []) :: [])`, the right-hand `[]` first gets type `'a3 list` which by unification with `int list list` yields the substitution

$$S_2(\texttt{'a3}) = \texttt{int list}.$$

Thus the type of `((5 :: []) :: [])` is `int list list`.

Returning to `(((5 :: []) :: []) :: [])`, the right-hand `[]` gets the type `'a4 list` which by unification with `int list list list` yields the substitution

$$S_3(\texttt{'a4}) = \texttt{int list list}$$

Thus the type of `(((5 :: []) :: []) :: [])` is `int list list list`.

Finally, returning to (2) and (3), we get to unify `'a1 list` with `int list list list`, yielding the substitution

$$S_4(\texttt{'a1}) = \texttt{int list list}.$$

The type of (2), and therefore the type of (1), is thus found to be `int list list list`.

Note that

```
[ [4] , [[5]] ]
```

is NOT well-typed. In an attempt to compute S_4, we would now be unifying `int list list` and `int list list list` and that gives a unification error.

To implement all this, we first extend the `TYPE` signature and introduce a new signature, `UNIFY`:

```
signature TYPE =
   sig
      eqtype tyvar
      val freshTyvar: unit -> tyvar
      ...
      val mkTypeTyvar: tyvar -> Type
          and unTypeTyvar: Type -> tyvar

      val mkTypeList: Type -> Type
          and unTypeList: Type -> Type
      type subst
      val Id: subst
                 (* the identify substitution;   *)
      val mkSubst: tyvar*Type -> subst
                 (* make singleton substitution; *)
      val on : subst * Type -> Type
                 (* application;                 *)

      val prType: Type->string          (* printing *)
   end


signature UNIFY=
   sig
      structure Type: TYPE
      exception NotImplemented of string
      exception Unify
      val unify: Type.Type * Type.Type -> Type.subst
   end;
```

The nice thing is that we can extend the typechecker without knowing anything about the inner workings of unification, simply by including a formal parameter of signature `UNIFY` in the typechecker functor:

```
functor TypeChecker
  (...
   structure Ty: TYPE
   structure Unify: UNIFY
    sharing Unify.Type = Ty
  )=
struct
  infix on
  val (op on) = Ty.on
  ...

  fun tc (exp: Ex.Expression): Ty.Type =
   (case exp of
      ...
    | Ex.LISTexpr [] =>
         let val new = Ty.freshTyvar ()
          in Ty.mkTypeList(Ty.mkTypeTyvar  new)
         end
    | Ex.CONSexpr(e1,e2) =>
```

```
        let val t1 = tc e1
            val t2 = tc e2
            val new = Ty.freshTyvar ()
            val newt= Ty.mkTypeTyvar new
            val t2' = Ty.mkTypeList newt
            val S1 = Unify.unify(t2, t2')
                     handle Unify.Unify=>
                     raise TypeError(e2,"expected list type")

            val S2 = Unify.unify(S1 on newt,S1 on t1)
                     handle Unify.Unify=>
                     raise TypeError(exp,
                      "element and list have different types")
         in S2 on (S1 on t2)
        end
    | ...

   )handle Unify.NotImplemented msg => raise NotImplemented msg

end; (*TypeChecker*)
```

We also have to extend the `Type` functor to meet the enriched `TYPE signature`. The easiest way of doing this is

```
functor Type():TYPE =
struct
  type tyvar = int
  val freshTyvar =
      let val r= ref 0 in fn()=>(r:= !r +1; !r) end
  datatype Type = INT
                | BOOL
                | LIST of Type
                | TYVAR of tyvar
  ...

  fun mkTypeTyvar tv = TYVAR tv
  and unTypeTyvar(TYVAR tv) = tv
    | unTypeTyvar _ = raise Type

  fun mkTypeList(t)=LIST t
  and unTypeList(LIST t)= t
    | unTypeList(_)= raise Type
  type subst = Type -> Type

  fun Id x = x

  fun mkSubst(tv,ty)=
     let fun su(TYVAR tv')= if tv=tv' then ty else TYVAR tv'
         |   su(INT) = INT
         |   su(BOOL)= BOOL
         |   su(LIST ty') = LIST (su ty')
      in su
     end

  fun on(S,t)= S(t)

  fun prType ...
  |   prType (LIST ty) = "(" ^ prType ty ^ ")list"
  |   prType (TYVAR tv) = "a" ^ makestring tv
end;
```

Exercise 4 Extend Version 2 to handle equality. All you have to do is to fill in the relevant case in the definition of the function `tc`. (See appendix B about how you get the source of Version 2).

4.3 VERSION 3: A different implementation of types

Version 3 arises from Version 2 by replacing the `Type` functor by a different implementation of types. The idea is that istead of having substitutions as functions, we can implement type variables by references (pointers) and then do substitutions directly by assignments.

In case you have not seen the reserved word `withtype` before, `withtype` is used to declare a type abbreviaton locally within a `datatype` declaration.

```
functor ImpType():TYPE =
struct
  datatype 'a option = NONE | SOME of 'a

  datatype Type = INT
                | BOOL
                | LIST of Type
                | TYVAR of tyvar

  withtype tyvar =  Type option ref
  type tyvar = Type option ref

  fun freshTyvar() = ref (NONE)

  exception Type

  fun mkTypeInt() = INT
  and unTypeInt(INT)=()
    | ...
    | unTypeInt(TYVAR(ref (SOME t)))= unTypeInt t
    | unTypeInt _ = raise Type



  ...
  type subst = unit

  val Id = ();

  exception MkSubst;

  fun mkSubst(tv,ty)=
     case tv of
       ref(NONE) => tv:= (SOME ty)
     | ref(SOME t) => raise MkSubst

  fun on(S,t)= t

  fun prType ...
  |   prType (TYVAR (ref NONE)) = "a?"
  |   prType (TYVAR (ref (SOME t))) = prType t
end;
```

We can now build two systems at the same time and compare the efficiency of the two implementations. The nice thing is that we do not have to modify the typechecker functor at all, nor do we even have to modify the unification functor; we can just extend the final sequence of structure declarations to use both implementations of types.

Exercise 5 When I did this, I found (to my surprise), that the functional version in some cases was twice as fast, and never slower than the imperative variant. The relative performance of the two vary greatly from expression to expression. Can you find an expression for which the imperative version really is faster? (See Appendix B for how to get hold of the source of Version 3). Be careful with generating very demanding tasks for the ML system; you can make it crash!

ML implementors normally opt for the imperative version. In all fairness, the above comparison ignores that composing substitutions is much easier in the imperative version than it is in the applicative version; in the fragment of Mini ML considered so far, we have not had to compose substitutions.

One should not be too concerned with performance issues at too early a stage. It can be surprisingly difficult to predict where efficiency is most needed, and it is much more important, at first, to get the overall structure of the system right. It was important, for example, that we did NOT make the constructors of the datatype `Type` visible in the signature `TYPE`, and that we wrote the unification algorithm in a way which does not use the internal structure of `Type`. Had we not done this, we would not have been able to switch from one implementation to another that easily, and therefore chances are that we would chosen the imperative one, assuming that it was the more efficient one, without ever trying the "obvious" applicative implentation.

4.4 VERSION 4: Introducing variables and `let`

We now extend Version 3 by implementing the type checking of `let` expressions and of identifiers.

The typechecker function `tc` now has to take TWO arguments,

```
tc(TE, e)
```

where `e` is an expression and `TE` is a TYPE ENVIRONMENT, which maps variables occurring free in `e` to TYPE SHEMES. The definition of what a type scheme is will be given below; for now it suffices to know that every type can be regarded as a type scheme.

To take an example, if `TE` maps `x` to `int` and `y` to `int`, then `tc` will deduce the type `int` for the expression `x+y`. (However, if `TE` mapped `y` to `bool`, there would be a type error.)

The fact that we can bind variables to expressions whose types have been inferred to contain type variables means that we get type variables in the type environment. For instance, to type check

```
    let x = [] in 4 :: x end
```

we first check `[]` yielding the type `'a1 list`, say. Then we bind `x` to the type scheme ∀`'a1.'a1 list`. Here the binding ∀`'a1` of `'a1` indicates that when we look up the the type of `x` in the type environment, we return a type obtained from the type scheme ∀`'a1.'a1 list` by instantiating the bound variables (here just `'a1`) by fresh type variables. In our example, when we look up `x` in the type environment during the checking of `4 :: x`, we instantiate `'a1` to a fresh type variable `'a2`, say, yielding the type `'a2 list` for `x`. Thus we get to unify `int list` against `'a2 list`, yielding the substitution of `int` for `'a2`.

Throughout the body of the `let`, `x` will be bound to ∀`'a1.'a1 list` in the type environment. Since we take a fresh instance of this type scheme each time we look up `x`, we can use `x` both as an `int list` and as an `int list list`, say:

```
    let x = [] in   (4::x)::x end
```

Exercise 6 Assuming that you instantiate the bound `'a1` to `'a3` when you meet the last occurrence of `x`, what two types should be unified, and what is the resulting substitution on `'a3` ?

The variable `x` is an example of POLYMORPHISM: after `x` has been declared, an occurrence of `x` can potentially be given infinitely many types: `int list`, `bool list`, `int list list`, and so on, all captured by the type scheme ∀ `'a1.'a1 list`. In ML, a TYPE SCHEME always takes the form $\forall\, \alpha_1 \cdots \alpha_n.\tau$, $(n \geq 0)$, where $\alpha_1, \ldots, \alpha_n$ are type variables and τ is a type. In the fragment of Mini ML considered so far, it will always be the case that any type variable occurring in τ is amongst the $\alpha_1, \ldots, \alpha_n$, but when one introduces functions and application, this no longer is the case.

Here is how we implement variables and `let`. We first extend the `TYPE` signature:

```
signature TYPE =
   sig
      ...
      type TypeScheme

      val instance: TypeScheme -> Type
      val close: Type -> TypeScheme

   end
```

Version 1 (Appendix A) already contains a signature for environments (find it). It was actually intended for the practical where you need it to extend the evaluator, but we can make use of it to implement type environments. The signature of the typechecker can be left unchanged, but we need to change the functor that builds the typechecker by including the environment management among the formal parameters:

```
functor TypeChecker
  (structure Ex: EXPRESSION
   structure Ty: TYPE
   structure Unify: UNIFY
      sharing Unify.Type = Ty
   structure TE: ENVIRONMENT
  )=
struct
  infix on
  val (op on) = Ty.on
  structure Exp = Ex
  structure Type = Ty
```

```
exception NotImplemented of string
exception TypeError of Ex.Expression * string

fun tc (TE: Ty.TypeScheme TE.Environment, exp: Ex.Expression): Ty.Type =
 (case exp of
    Ex.BOOLexpr b => Ty.mkTypeBool()
  | Ex.NUMBERexpr _ => Ty.mkTypeInt()
  | Ex.SUMexpr(e1,e2)  => checkIntBin(TE,e1,e2)
  | Ex.DIFFexpr(e1,e2) => checkIntBin(TE,e1,e2)
  | Ex.PRODexpr(e1,e2) => checkIntBin(TE,e1,e2)
  | Ex.LISTexpr [] =>
       let val new = Ty.freshTyvar ()
        in Ty.mkTypeList(Ty.mkTypeTyvar  new)
       end
  | Ex.LISTexpr(e::es) => tc (TE, Ex.CONSexpr(e,Ex.LISTexpr es))
  | Ex.CONSexpr(e1,e2) =>
      let val t1 = tc(TE, e1)
          val t2 = tc(TE, e2)
          val new = Ty.freshTyvar ()
          val newt= Ty.mkTypeTyvar new
          val t2' = Ty.mkTypeList newt
          val S1 = Unify.unify(t2, t2')
                   handle Unify.Unify=>
                   raise TypeError(e2,"expected list type")

          val S2 = Unify.unify(S1 on newt,S1 on t1)
                   handle Unify.Unify=>
                   raise TypeError(exp,"element and list have different types")
       in S2 on (S1 on t2)
      end
  | Ex.EQexpr _ => raise NotImplemented "(equality)"
  | Ex.CONDexpr _ => raise NotImplemented "(conditional)"
  | Ex.DECLexpr(x,e1,e2) =>
       let val t1 = tc(TE,e1);
           val typeScheme = Ty.close(t1)
        in tc(TE.declare(x,typeScheme,TE), e2)
       end
  | Ex.RECDECLexpr _ => raise NotImplemented "(rec decl)"
  | Ex.IDENTexpr x   =>
       (Ty.instance(TE.retrieve(x,TE))
       handle TE.Retrieve _ =>
        raise TypeError(exp,"identifier " ^ x ^ " not declared"))
  | Ex.LAMBDAexpr _  => raise NotImplemented "(function)"
```

```
    | Ex.APPLexpr _ => raise NotImplemented    "(application)"

   )handle Unify.NotImplemented msg => raise NotImplemented msg

  and checkIntBin(TE,e1,e2) =
    let val t1 = tc(TE,e1)
        val _  = Ty.unTypeInt t1
                 handle Ty.Type=> raise TypeError(e1,"expected int")
        val t2 = tc(TE,e2)
        val _  = Ty.unTypeInt t2
                 handle Ty.Type=> raise TypeError(e2,"expected int")
     in Ty.mkTypeInt()
    end;

  fun typecheck(e) = tc(TE.emptyEnv,e)

end; (*TypeChecker*)
```

Then we extend the Type functor to match the TYPE signature:

```
functor Type():TYPE =
struct
  ...
  datatype TypeScheme = FORALL of tyvar list * Type

  fun instance(FORALL(tyvars,ty))=
  let val old_to_new_tyvars = map (fn tv=>(tv,freshTyvar())) tyvars
      exception Find;
      fun find(tv,[])= raise Find
      |   find(tv,(tv',new_tv)::rest)=
          if tv=tv' then new_tv else find(tv,rest)
      fun ty_instance INT = INT
      |   ty_instance BOOL = BOOL
      |   ty_instance (LIST t) = LIST(ty_instance t)
      |   ty_instance (TYVAR tv) =
             TYVAR(find(tv,old_to_new_tyvars)
                   handle Find=> tv)

  in
     ty_instance ty
  end
  fun close(ty)=
  let fun fv(INT) = []
      |   fv(BOOL)= []
      |   fv(LIST t) = fv(t)
      |   fv(TYVAR tv) = [tv]
   in FORALL(fv ty,ty)
  end

end;
```

Finally, the system is re-built as in Version 2, except that we have to provide and link in an `Environment` functor which matches `ENVIRONMENT`.

Exercise 7 Extend Version 4 with `if .. then .. else`. (This extension has no subtle implications for the type checking.)

Exercise 8 [For the extra keen] Extend Version 4 to cope with lambda abstraction (`fn`) and application. First, you have to introduce arrow types with constructors and destructors. Then you have to change the type of `close` so that it takes two arguments, namely a type environment and a type. It should return the type scheme that is obtained by quantifying on all the type variables that occur in the type but do not occur free in the type environment.

Then you can modify the type checker. When you type check a lambda abstraction, you just bind the formal parameter to the trivial type scheme which is just a fresh type variable (no quantified variables). Thus the type environment can now contain type schemes with free type variables.

An application `tc(TE,e)` now yields two arguments, namely a type t and a substitution S; the idea is that if you apply the substitution S to the type environment `TE`, which now can contain free type variables, the expression `e` has the type t. When an expression consists of more than one subexpression, the type environment gradually becomes more and more specific by applying the substitutions produced by the checking of the subexpressions one by one. Moreover, the substitution returned from the whole expression is the composition of these individual substitutions. (You have to extend the `TYPE` signature (and the `Type` functor) with composition of substitutions.

Finally, you can extend the unification algorithm to cope with arrow types. (This will also use composition of substitutions.)

4.5 Acknowledgement

The parser and evaluator and all the signatures related to them are due to Nick Rothwell.

Appendix A: The bare Interpreter

```
(* interp1.sml : VERSION 1: the bare interpreter *)

signature INTERPRETER=
   sig
      val interpret: string -> string
      val eval: bool ref
      and tc  : bool ref
   end;

                  (* syntax *)

signature EXPRESSION =
   sig
      datatype Expression =
         SUMexpr of Expression * Expression   |
         DIFFexpr of Expression * Expression   |
         PRODexpr of Expression * Expression   |
         BOOLexpr of bool   |
         EQexpr of Expression * Expression   |
         CONDexpr of Expression * Expression * Expression   |
         CONSexpr of Expression * Expression   |
         LISTexpr of Expression list   |
         DECLexpr of string * Expression * Expression   |
         RECDECLexpr of string * Expression * Expression   |
         IDENTexpr of string   |
         LAMBDAexpr of string * Expression   |
         APPLexpr of Expression * Expression   |
         NUMBERexpr of int
   end


              (* parsing *)

signature PARSER =
   sig
      structure E: EXPRESSION

      exception Lexical of string
      exception Syntax of string
      val parse: string -> E.Expression
   end


                        (* environments *)

signature ENVIRONMENT =
   sig
      type 'object Environment

      exception Retrieve of string

      val emptyEnv: 'object Environment
      val declare: string * 'object * 'object Environment
          -> 'object Environment
      val retrieve: string * 'object Environment -> 'object
   end

                        (* evaluation *)
signature VALUE =
   sig
      type Value
      exception Value

      val mkValueNumber: int -> Value
          and unValueNumber: Value -> int

      val mkValueBool: bool -> Value
          and unValueBool: Value -> bool

      val ValueNil: Value
      val mkValueCons: Value * Value -> Value
          and unValueHead: Value -> Value
          and unValueTail: Value -> Value

      val eqValue: Value * Value -> bool
      val printValue: Value -> string
   end


signature EVALUATOR =
   sig
```

```
      structure Exp: EXPRESSION
      structure Val: VALUE
      exception Unimplemented
      val evaluate: Exp.Expression -> Val.Value
   end

                  (* type checking *)
signature TYPE =
   sig
      type Type

(*constructors and decstructors*)
      exception Type
      val mkTypeInt: unit -> Type
          and unTypeInt: Type -> unit

      val mkTypeBool: unit -> Type
          and unTypeBool: Type -> unit

      val prType: Type->string
   end



signature TYPECHECKER =
   sig
      structure Exp: EXPRESSION
      structure Type: TYPE
      exception NotImplemented of string
      exception TypeError of Exp.Expression * string
      val typecheck: Exp.Expression -> Type.Type
   end;

                  (* the interpreter*)

functor Interpreter
   (structure Ty: TYPE
    structure Value : VALUE
    structure Parser: PARSER
    structure TyCh: TYPECHECKER
    structure Evaluator:EVALUATOR
      sharing Parser.E = TyCh.Exp = Evaluator.Exp
          and TyCh.Type = Ty
          and Evaluator.Val = Value
   ): INTERPRETER=

struct
  val eval= ref true    (* toggle for evaluation *)
  and tc  = ref true    (* toggle for type checking *)
  fun interpret(str)=
    let val abstsyn= Parser.parse str
        val typestr= if !tc then
                     Ty.prType(TyCh.typecheck abstsyn)
                     else "(disabled)"
        val valuestr= if !eval then
                  Value.printValue(Evaluator.evaluate abstsyn)
                      else "(disabled)"

    in  valuestr ^ " : " ^ typestr
    end
    handle Evaluator.Unimplemented =>
                "Evaluator not fully implemented"
         | TyCh.NotImplemented msg =>
                "Typechecker not fully implemented " ^ msg
         | Value.Value   => "Run-time error"
         | Parser.Syntax msg => "Syntax Error: " ^ msg
         | Parser.Lexical msg=> "Lexical Error: " ^ msg
         | TyCh.TypeError(_,msg)=> "Type Error: " ^ msg
end;

                    (* the evaluator *)

functor Evaluator
  (structure Expression: EXPRESSION
   structure Value: VALUE):EVALUATOR=

   struct
      structure Exp= Expression
      structure Val= Value
      exception Unimplemented

      local
         open Expression Value
         fun evaluate exp =
            case exp
              of BOOLexpr b => mkValueBool b
            | NUMBERexpr i => mkValueNumber i
            | SUMexpr(e1, e2) =>
                let val e1' = evaluate e1
                    val e2' = evaluate e2
                in
                   mkValueNumber(unValueNumber e1' +
                                 unValueNumber e2')
                end

            | DIFFexpr(e1, e2) =>
                let val e1' = evaluate e1
                    val e2' = evaluate e2
                in
                   mkValueNumber(unValueNumber e1' -
                                 unValueNumber e2')
                end

            | PRODexpr(e1, e2) =>
                let val e1' = evaluate e1
                    val e2' = evaluate e2
                in
                   mkValueNumber(unValueNumber e1' *
                                 unValueNumber e2')
                end

            | EQexpr _ => raise Unimplemented
            | CONDexpr _ => raise Unimplemented
            | CONSexpr _ => raise Unimplemented
            | LISTexpr _ => raise Unimplemented
            | DECLexpr _ => raise Unimplemented
            | RECDECLexpr _ => raise Unimplemented
            | IDENTexpr _ => raise Unimplemented
            | LAMBDAexpr _ => raise Unimplemented
            | APPLexpr _ => raise Unimplemented

      in
         val evaluate = evaluate
      end
   end;

                     (* the typechecker *)

functor TypeChecker
```

```
  (structure Ex: EXPRESSION
   structure Ty: TYPE)=
struct
  structure Exp = Ex
  structure Type = Ty
  exception NotImplemented of string
  exception TypeError of Ex.Expression * string

  fun tc (exp: Ex.Expression): Ty.Type =
    case exp of
      Ex.BOOLexpr b => raise NotImplemented
                         "(boolean constants)"
    | Ex.NUMBERexpr _ => Ty.mkTypeInt()
    | Ex.SUMexpr(e1,e2)  => checkIntBin(e1,e2)
    | Ex.DIFFexpr _ => raise NotImplemented "(minus)"
    | Ex.PRODexpr _ => raise NotImplemented "(product)"
    | Ex.LISTexpr _ => raise NotImplemented "(lists)"
    | Ex.CONSexpr _ => raise NotImplemented "(lists)"
    | Ex.EQexpr _ => raise NotImplemented "(equality)"
    | Ex.CONDexpr _ => raise NotImplemented "(conditional)"
    | Ex.DECLexpr _ => raise NotImplemented "(declaration)"
    | Ex.RECDECLexpr _ => raise NotImplemented "(rec decl)"
    | Ex.IDENTexpr _   => raise NotImplemented "(identifier)"
    | Ex.LAMBDAexpr _  => raise NotImplemented "(function)"
    | Ex.APPLexpr _ => raise NotImplemented "(application)"

  and checkIntBin(e1,e2) =
    let val t1 = tc e1
        val _  = Ty.unTypeInt t1
                 handle Ty.Type=>
                 raise TypeError(e1,"expected int")
        val t2 = tc e2
        val _  = Ty.unTypeInt t2
                 handle Ty.Type=>
                 raise TypeError(e2,"expected int")
     in Ty.mkTypeInt()
    end;

  val typecheck = tc

end; (*TypeChecker*)
```

```
                    (* the basics -- nullary functors *)

functor Type():TYPE =
struct
  datatype Type = INT
                | BOOL

  exception Type

  fun mkTypeInt() = INT
  and unTypeInt(INT)=()
    | unTypeInt(_)= raise Type

  fun mkTypeBool() = BOOL
  and unTypeBool(BOOL)=()
    | unTypeBool(_)= raise Type

  fun prType INT = "int"
  |   prType BOOL= "bool"
end;



functor Expression(): EXPRESSION =
   struct
      type 'a pair = 'a * 'a

      datatype Expression =
         SUMexpr of Expression pair   |
         DIFFexpr of Expression pair   |
         PRODexpr of Expression pair   |
         BOOLexpr of bool   |
         EQexpr of Expression pair   |
         CONDexpr of Expression * Expression * Expression   |
         CONSexpr of Expression pair   |
         LISTexpr of Expression list   |
         DECLexpr of string * Expression * Expression   |
         RECDECLexpr of string * Expression * Expression   |
         IDENTexpr of string   |
         LAMBDAexpr of string * Expression   |
         APPLexpr of Expression * Expression   |
         NUMBERexpr of int
   end;

functor Value(): VALUE =
   struct
      type 'a pair = 'a * 'a

      datatype Value = NUMBERvalue of int   |
                       BOOLvalue of bool   |
                       NILvalue   |
                       CONSvalue of Value pair

      exception Value

      val mkValueNumber = NUMBERvalue
      val mkValueBool = BOOLvalue

      val ValueNil = NILvalue
      val mkValueCons = CONSvalue

      fun unValueNumber(NUMBERvalue(i)) = i   |
          unValueNumber(_) = raise Value

      fun unValueBool(BOOLvalue(b)) = b   |
          unValueBool(_) = raise Value

      fun unValueHead(CONSvalue(c, _)) = c   |
          unValueHead(_) = raise Value

      fun unValueTail(CONSvalue(_, c)) = c   |
          unValueTail(_) = raise Value

      fun eqValue(c1, c2) = (c1 = c2)

(* Pretty-printing *)
      fun printValue(NUMBERvalue(i)) = makestring(i)   |
          printValue(BOOLvalue(true)) = "true"   |
          printValue(BOOLvalue(false)) = "false"   |
          printValue(NILvalue) = "[]"   |
          printValue(CONSvalue(cons)) = "[" ^
                 printValueList(cons) ^ "]"
          and printValueList(hd, NILvalue) = printValue(hd) |
              printValueList(hd, CONSvalue(tl)) =
                 printValue(hd) ^ ", " ^ printValueList(tl) |
              printValueList(_) = raise Value
end;
```

Appendix B: Files

The following files are available in the directory `/usr/cheops/mads/course`

- `interp1.sml`   Version 1 (as included in Appendix A).
- `interp2.sml` ··· `interp4.sml`   The other versions.
- `build1.sml`   the structure declarations needed to build Version 1.
- `build2.sml` ··· `build4.sml`   Similarly for the other versions.
- `parser.sml`   The parser functor.

To build Version 3, say, you type the following (assuming you have copied the files to your directory):

```
use "interp3.sml";
use "parser.sml";
use "build3.sml";
```

Since the parser functor is completely closed, you dont have to include it more than once in every session, although you will probably want to build your system several times while you experiment with the extensions.